SU(N) for large N in the Hamiltonian Formalism
in 2+1 dimensions

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What is new in our work?

- Improved Hamiltonians, generated systematically and independently of the (improved) action approach
- We do the relevant integrals over the SU(N) Haar measure analytically, so we get to MUCH larger $N$ ($N \leq 25)$
- We have indications that the glueball mass spectrum in 2+1 dimensions in the large $N$ limit is that of an harmonic oscillator in 2 dimensions.
Improving the Hamiltonian

The Kogut-Susskind Hamiltonian for pure glue SU($N$) gauge theory in temporal gauge is $H^{(0)} = K^{(0)} + V^{(0)}$, where the kinetic and potential terms are given respectively by,

$$K^{(0)} = \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ E^L_i(x) E^L_i(x) \right\} \quad V^{(0)} = -\frac{2N}{ag^2} \sum_{x,i<j} P_{ij}(x).$$

Here $E^L$ is the lattice chromo-electric field, $N$ is the dimension of the gauge group, and $P_{ij}(x)$ is the plaquette operator,

$$P_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \left\{ \begin{array}{c} \square \\ \square \end{array} \right\}.$$

We make improvements by reducing the errors between the lattice Hamiltonian and the continuum Hamiltonian.
Improved Hamiltonians — Classical Improvement of the potential term

Improvement to $O(a^2)$ is achieved by introducing rectangular loops into the potential term. The improved potential is

$$V^{(1)} = -\frac{2N}{ag^2} \sum_{x,i<j} \left[ XP_{ij}(x) + \frac{Y}{2} \left( R_{ij}(x) + R_{ji}(x) \right) \right],$$

where the rectangle operator with the long side in the $i$ direction is

$$R_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \left\{ \begin{array}{c} \square \\ \end{array} \right\}$$

and the constants $X$ and $Y$ are fixed so that the order $a^2$ errors vanish. i.e $X = 5/3$ and $Y = -1/6$, and

$$V^{(1)} = -\frac{2N}{ag^2} \sum_{x,i<j} \left[ \frac{5}{3} P_{ij}(x) - \frac{1}{12} \left( R_{ij}(x) + R_{ji}(x) \right) \right]$$

This can be extended to higher order as for actions if necessary.
Improved Hamiltonians — Classical Corrections to the Kinetic Term — Nearest Neighbours

The Kinetic Term also contains errors, which arise because the lattice electric field is not the time derivative of the lattice gauge field. The canonical commutation relations require differences of $\mathcal{O}(a^2)$ in the lattice electric field. To preserve manifest gauge invariance add in additional terms which are gauge invariant, and adjust the coefficients to cancel the $O(a^2)$ corrections, thus generating an improved kinetic term.

The nearest neighbour correction is

$$K^{(1)}_{(1)} = \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ \frac{5}{6} E_i^L(x) E_i^L(x) + \frac{1}{6} E_i^L(x) U_i(x) E_i^L(x + a) U_i^\dagger(x) \right\}.$$  

which now has $\mathcal{O}(a^4)$ corrections. It is relatively straightforward to cancel these too, should it be necessary.
The Tadpole Improved Hamiltonian

In the action formulation tadpole improvement is implemented by dividing all link operators by the mean link $u_0$.

BUT

Under tadpole improvement the electric field does not change.

Including tadpole improvement leads to the simplest and most accurate order $a^2$ tadpole improved Hamiltonian:

$$H^{(1)} = K^{(1)} + V^{(1)}$$

$$= \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ \frac{5}{6} E^L_i(x) E^L_i(x) + \frac{1}{6u_0^2} E^L_i(x) U(x) E^L_i(x + ai) U^\dagger(x) \right\}$$

$$- \frac{2N}{ag^2} \sum_{x,i<j} \left[ \frac{5}{3u_0^4} P_{ij}(x) - \frac{1}{12u_0^6} \left( R_{ij}(x) + R_{ji}(x) \right) \right].$$

which we use as the basis for our calculations.
The Ground State

Our trial state is the one plaquette trial state, first suggested by Greensite:

$$|\phi_0\rangle = \exp\left[\frac{\kappa}{2} \sum_p (Z_p + \bar{Z}_p)\right] |0\rangle.$$  

$|0\rangle$ is the strong coupling vacuum The $p$-sum extends over all plaquettes on the lattice and $Z_p = \text{Tr} \left[ U_i(\vec{x}) U_j(\vec{x} + \vec{i}a) U^\dagger_i(\vec{x} + \vec{j}a) U^\dagger_j(\vec{x}) \right]$, is the plaquette variable. $\bar{Z}_p$ is the trace of the link operators in the opposite direction. The vacuum energy density is

$$\epsilon_0 = \frac{a}{N_p} \langle \bar{H} \rangle$$

$$= \left[ (1 - \kappa) \left( \frac{N^2 - 1}{2\beta} \right) e - \frac{2\beta(1 + 4\kappa)}{N u_0^4} \right] \langle \square \rangle + \frac{2\kappa\beta}{N u_0^6} \langle \square \rangle + 2(1 + 3\kappa)\beta.$$  

$N_p$ is the number of plaquettes on the lattice and $\beta = N/g^2$ is the inverse coupling, $\kappa = 0, 1/6, 1/(6u_0)$ for K-S, improved and tadpole improved.
Variational calculation
Do a simple variational calculation using Green-site’s trial state $|\phi_0\rangle = \exp\left[\frac{c}{2} \sum_p (Z_p + \bar{Z}_p)\right] |0\rangle$.

generated from the strong coupling vacuum, and look at the specific heat. Tadpole improvement does improve
ie it pushes the peak in $C_V$, the transition from strong to weak coupling to smaller $\beta$, or larger $g$. 

The SU(2) result

![Graph showing specific heat $C_V$ vs $\beta$. The graph compares unimproved and tadpole-improved cases. The peak of the unimproved curve is shifted to the right compared to the tadpole-improved curve.]}
Integrals

In 2+1 dimensions the variables can be changed from links to plaquettes with unit Jacobian. The plaquettes then become independent variables.

\[ G_{SU(N)}(c, d) = \int dU_p e^{cZ_p + d\bar{Z}_p} \]

and similar generating functions can generate the integrals we need to perform variational calculations in 2+1 dimensions.

**These integrals can be done analytically**
The SU(N) integral

Start with the U(N) integral, which can be done by transforming to the diagonal basis, and gives $G_{U(N)}(c, d)$ as a Toeplitz determinant

$$G_{U(N)}(c, d) = \det \left[ I_{j-i} \left( 2\sqrt{cd} \right) \right]_{1 \leq i,j \leq N}.$$

To calculate the SU(N) result the restriction $\det U = 1$ must be built into the integration measure, which can be easily done in the diagonal basis. Then, with

$$g_n(c, d) = \int_0^{2\pi} \frac{dx}{2\pi} \exp(\imath nx + ce^{i\imath x} + de^{-i\imath x}) = \left( \frac{d}{c} \right)^{n/2} I_n \left( 2\sqrt{cd} \right),$$

$$G_{SU(N)}(c, d) = \int_{SU(N)} dU e^{\imath \text{Tr} U + \text{dTr} U^\dagger} = \sum_{m=-\infty}^{\infty} \det \left[ g_{m+j-i}(c, d) \right]_{1 \leq i,j \leq N} = \sum_{l=-\infty}^{\infty} \left( \frac{d}{c} \right)^{lN/2} \det \left[ I_{l+j-i} \left( 2\sqrt{cd} \right) \right]_{i \leq i,j \leq N}. $$
Variational Coefficient as a function of $\beta$
Wrong behaviour??

Note that $c(\beta) \propto \beta$ for large $\beta$.

BUT to obtain the correct continuum limit, one must have

$$c(\beta) \propto \beta^2 \quad \text{as} \quad \beta \to \infty$$

for $\text{SU}(2)$ \quad $c(\beta) \to \frac{0.81 \pm 0.02}{g^4}$ \quad as \quad $\beta \to \infty$

It will turn out that this apparently wrong behaviour is not a problem when we try to estimate the energies of the excited states, and work in a scaling region, as we do now.
A basis for excited states

Use all rectangles which fit in a square of side $L_{max}$ to form the basis. $w_l(\vec{x})$ is one of the rectangular Wilson loops with $\vec{x}$ as the bottom left corner. $l$ labels the possible rectangles.

Then define $W_l(\vec{x}) = \text{Tr} \left[ w_l(\vec{x}) \pm w_l^\dagger(\vec{x}) \right]$ for the symmetric sector and antisymmetric sector, and introduce the basis of excited states

$$|l\rangle = \sum_{\vec{x}} \left[ W_l(\vec{x}) - \langle W_l(\vec{x}) \rangle \right] |\phi_0\rangle.$$  

and minimise the expectation value of $H$ in the states generated from this basis to get estimates of the excited state energies.

We expect the dimensionless massgap $a\Delta m/g^2$ to become constant in the scaling region.
(a) Symmetric SU(2) massgap

(b) Symmetric SU(3) massgap

(c) Symmetric SU(4) massgap

(d) Symmetric SU(5) massgap

Lowest symmetric mass gap for SU(N), N \in (2, 5)
Comparison to other results

**SU(2)**

For the unimproved SU(2) case, the lowest two eigenstates agree precisely with the calculation of Arisue. This serves as a check on our counting in calculating the possible overlaps of excited states. Our calculation is in disagreement with that of Arisue at the 3rd eigenstate. It appears that our 4th eigenstate corresponds to Arisue’s 3rd and that our third eigenstate does not appear in his results. The reasons for this are not clear.

**SU(3), SU(4), SU(5)**

For the other $N$ our scaling region corresponds to the second excited state $O^{++*}$ of Teper, rather than the lowest $0^{++}$. The latter corresponds to the position of the minimum in the curves.

How should we understand this?

Return to the dependence of the variational parameter on $\beta$. 

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Scaling Regions

\[ E[\phi] = \frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle} \] is the energy functional for some state \( |\phi\rangle \).

\[ |\phi_0\rangle \equiv |\phi_0(c(\beta))\rangle = \exp \left[ c(\beta) \sum_p (U_p + U_p^*) \right] |0\rangle \]

is an approximation to the ground state.

\[ |\phi_1\rangle \equiv |\phi_1(c(\beta), d_1(\beta), \ldots, d_n(\beta))\rangle \]

is chosen to be orthogonal to both the exact and approximate ground states and approximates an excited state of a particular symmetry. The dimensionless mass gap

\[ \frac{a}{g^2} \Delta m \equiv \mu(c(\beta), d_1(\beta), \ldots, d_n(\beta)) = \Delta m_0 = \text{const.} \quad \forall \beta \in S. \]

The variational parameters may be rescaled with respect to \( \beta \) and the same constant result is obtained but for a different range of couplings \( T \):

\[ \mu(c(f(\beta)), d_1(f(\beta)), \ldots, d_n(f(\beta))) = \Delta m_0 \quad \forall \beta \in TT = f^{-1}(S). \]

Choose \( f(\beta) \) such that \( c(f(\beta)) \) gives the correct form for the ground state parameter in the scaling region. This is why we get the same results as Arisue, with an apparently wrong continuum limit for \( c(\beta) \)
Large N behaviour of the dip

This transformation of the scaling region could be used to transform the minima seen in the curves for $N \geq 3$ into a long flat scaling region. So it is legitimate to regard it as a mass gap. Our simple calculations are giving two scaling regions. We don’t yet understand how or why this happens. But we regard the symmetric dip as the lowest $0^{++}$ state.
\[ \frac{a \Delta M^c}{N g^2} = \gamma_1 + \frac{\gamma_2}{N^2 \gamma_3} \]

for \( N \geq 8 \) gives best fit parameters

\[ \gamma_1 = 0.83262 \pm 0.00022 \]
\[ \gamma_2 = -0.97 \pm 0.13 \]
\[ \gamma_3 = 0.990 \pm 0.035. \]

for \( N \geq 8 \) gives the expected \( N^{-2} \) behaviour.

\[ \frac{a \Delta M^c}{N g^2} = \gamma_1 + \frac{\gamma_2}{N^2} + \frac{\gamma_3}{N^2 \gamma_4}. \]

The best fit parameters when fit on the whole data set are

\[ \gamma_1 = 0.83287 \pm 0.00011 \]
\[ \gamma_2 = -1.116 \pm 0.028 \]
\[ \gamma_3 = 1.715 \pm 0.085 \]
\[ \gamma_4 = 1.627 \pm 0.050 \]

but the \( N^{-2}, N^{-4} \) fit is almost as good.
Comparison to Lucini and Teper Monte-Carlo results

Our best estimate of the $N \to \infty$ limit of $a \Delta M^c/(Ng^2)$ is $0.83256 \pm 0.00007$.

Lucini and Teper give the $N \to \infty$ limit of the lowest $0^{++}$ glueball mass as $0.8116 \pm 0.0036$ based on a quadratic extrapolation to the $N \to \infty$ limit of $2 \leq N \leq 6$ data.

The $N \to \infty$ limits agree, but the coefficients of the $N^{-2}$ term differ

Us: $-(0.97 \pm 0.13)/N^2$  \hspace{1cm} Lucini and Teper: $-(0.090 \pm 0.028)/N^2$,

This is not the result of the small number of $N$ used in the extrapolation, it could be because the couplings differ. Should be followed up.
Extrapolating the high beta plateau

Graph 1: A plot showing the behavior of $\frac{g_{\text{eff}}}{g}$ as a function of $\frac{1}{g^2}$ for SU(4), SU(6), SU(8), SU(10), SU(11), SU(12), SU(13), SU(14), and SU(15).

Graph 2: A plot showing the behavior of $\frac{aM_{\text{eff}}}{g}$ as a function of $\frac{1}{N^2}$ with a dashed line representing $\kappa_1^{++}(N^2)$. 
Compare to Lucini and Teper

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<th>Our results</th>
<th>Lucini &amp; Teper</th>
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<td>6</td>
<td>$3.7667 \pm 0.0013$</td>
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There is a hint of a small plateau in the level 3 plot, which could be the missing $0^{++}$ state.

We have problems with some low lying states, maybe we need more complicated lattice animals in the prefactors (not too hard) or in the exponential (back to Monte Carlo)
The $N = \infty$ spectrum

$$m_n(0^{++}) = (0.256 \pm 0.002)(2n + 0 + 1)$$
$$m_n(0^{--}) = (0.151 \pm 0.002)(2n + 4 + 1)$$
$$m_n(2^{--}) = (0.1495 \pm 0.0008)(2n + 6 + 1).$$

We notice a similarity with the two dimensional harmonic oscillator spectrum,

$$E_n \propto 2n + J + 1,$$

Model:

$$m_n(J^{PC}) = \gamma_{PC} \left[ 2n + \gamma(J^{PC}) + 1 \right],$$

where $\gamma_{PC}$ — spin independent and

$$\gamma(J^{PC}) = J \mod 4.$$

We have $\gamma_{--} \approx 0.15$
and $\gamma_{++} = 0.256 \pm 0.002$. 
I leave you to puzzle over whether this apparent harmonic oscillator behaviour means anything or not!

Further details in
hep-lat/0303018
hep-lat/0303022